Vector-Valued Functions

Definition A vector-valued function, or vector function

$$r : I \subset \mathbb{R} \to \mathbb{R}^n$$

is a function whose domain is a subset I of $\mathbb{R}$ and whose range is a set of vectors in $\mathbb{R}^n$, that is

$$r(t) = \langle r_1(t), r_2(t), \ldots, r_n(t) \rangle = \sum_{i=1}^{n} r_i(t) e_i \in \mathbb{R}^n \quad \text{for each } t \in I,$$

where

- $e_i = (0, \ldots, 0, \overset{i^{\text{th}}}{1}, 0, \ldots, 0)$ is the unit vector in the positive i^{th} coordinate,
- $r_1, r_2, \ldots, r_n : I \to \mathbb{R}$ are called the component functions of r.

Definitions Let $r : I \subset \mathbb{R} \to \mathbb{R}^n$ be a vector function defined by

$$r(t) = \langle r_1(t), r_2(t), \ldots, r_n(t) \rangle = \sum_{i=1}^{n} r_i(t) e_i \quad t \in I.$$

- $\lim_{t \to a} r(t)$ exists if and only if $\lim_{t \to a} r_i(t)$ exists for each $1 \le i \le n$.
  Furthermore, if $\lim_{t \to a} r(t)$ exists, then

  $$\lim_{t \to a} r(t) = \langle \lim_{t \to a} r_1(t), \lim_{t \to a} r_2(t), \ldots, \lim_{t \to a} r_n(t) \rangle.$$

- r is continuous at $t = a$ if

  $$\lim_{t \to a} r(t) = r(a).$$

  Thus r is continuous at $t = a$ if and only if its component functions $r_1, r_2, \ldots, r_n$ are continuous at a.
- r is differentiable at $t = a$ if and only if its component functions $r_1, r_2, \ldots, r_n$ are differentiable at a.
  Furthermore, if r is differentiable at $t = a$, then its derivative $r'(a)$ is a vector in $\mathbb{R}^n$ defined by

  $$r'(a) = \lim_{t \to a} \frac{r(t) - r(a)}{t - a} = \langle \lim_{t \to a} \frac{r_1(t) - r_1(a)}{t - a}, \lim_{t \to a} \frac{r_2(t) - r_2(a)}{t - a}, \ldots, \lim_{t \to a} \frac{r_n(t) - r_n(a)}{t - a} \rangle = \sum_{i=1}^{n} r_i'(a) e_i.$$

- r is integrable on $[a, b] \subset I$ if and only if its component functions $r_1, r_2, \ldots, r_n$ are integrable on $[a, b]$.

  Furthermore, if r is integrable on $[a, b]$, then its definite integral $\int_a^b r(t)\, dt$ is a vector in $\mathbb{R}^n$ defined by

  $$\int_a^b r(t)\, dt = \langle \int_a^b r_1(t)\, dt, \int_a^b r_2(t)\, dt, \ldots, \int_a^b r_n(t)\, dt \rangle = \sum_{i=1}^{n} \left[\int_a^b r_i(t)\, dt \right] e_i.$$

Definition If $r : I \subset \mathbb{R} \to \mathbb{R}^3$ is continuous on an interval I, then the set C defined by

$$C = \{(x_1, x_2, x_3) \mid x_1 = r_1(t),\ x_2 = r_2(t),\ x_3 = r_3(t),\ t \in I\} = \text{trace of } r \text{ throughout } I$$

is called a space curve given by the vector function (or parametrization) $r(t)$, $t \in I$.

Note that a single curve C can be represented by more than one vector function. For instance,

$$r_1(t) = \langle t, t^2, t^3 \rangle, 1 \le t \le 2 \quad \text{and} \quad r_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle, 0 \le u \le \ln 2$$

are parametrizations of the same twisted cubic curve, and the connection between the parameters t and u is given by $t = e^u$.

Examples

1. $r(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$ is a vector function from $[0, 3)$ to $\mathbb{R}^3$.
2. Find $\lim_{t \to 0} r(t)$, wehre $r(t) = (1+t^3)\mathbf{i} + te^{-t}\mathbf{j} + \frac{\sin t}{t}\mathbf{k}$, where $\mathbf{i} = e_1, \mathbf{j} = e_2, \mathbf{k} = e_3$.
3. Sketch the curve whose vector equation is $r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $t \in \mathbb{R}$. Note that this curve is called a **helix**.
4. Find the derivative of $r(t)$ at $t = 0$, wehre $r(t) = (1+t^3)\mathbf{i} + te^{-t}\mathbf{j} + \frac{\sin t}{t}\mathbf{k}$, and find the unit tangent vector at the point where $t = 0$.
5. Find $\int_0^{\pi/2} r(t)\, dt$, where $r(t) = 2\cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$.

Theorem Let I be an interval, $u, v : I \to \mathbb{R}^3$ be differentiable vector functions, c be a scalar, $f : I \to \mathbb{R}$ be a differentiable function and $g : I \to I$ be a differentiable function. Then

1. $\frac{d}{dt}[u(t) + v(t)] = u'(t) + v'(t)$
2. $\frac{d}{dt}[cu(t)] = cu'(t)$
3. $\frac{d}{dt}[f(t)u(t)] = f'(t)u(t) + f(t)u'(t)$
4. $\frac{d}{dt}[u(t) \cdot v(t)] = u'(t) \cdot v(t) + u(t) \cdot v'(t)$,
5. $\frac{d}{dt}[u(t) \times v(t)] = u'(t) \times v(t) + u(t) \times v'(t)$,
6. $\frac{d}{dt}[u(g(t))] = g'(t)u'(g(t))$ (Chain Rule)

where $\langle u(t) \cdot v(t) \rangle = u(t) \cdot v(t)$ and $u(t) \times v(t)$ denotes the inner product and the cross productof $u(t)$ and $v(t)$, respectively.

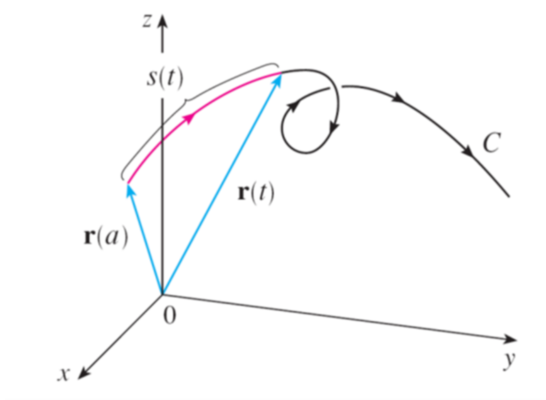
Theorem If $r : I \to \mathbb{R}^3$ is differentiable and $|r(t)| = c$ (a constant), then $r'(t)$ is orthogonal to $r(t)$ for all t.

Arc Length Let C be a curve given by a vector equation (or vector function, or parametrization)

$$r(t) = \langle r_1(t), r_2(t), r_3(t) \rangle = r_1(t)\mathbf{i} + r_2(t)\mathbf{j} + r_3(t)\mathbf{k}, \quad t \in [a, b]$$

where r' **is continuous and** C **is traversed exactly once as** t **increases from** a **to** b. Then the length of C is

$$L(C) = \int_a^b |r'(t)|\, dt = \int_a^b \sqrt{\sum_{i=1}^{3} [r_i'(t)]^2}\, dt = \int_a^b \sqrt{\sum_{i=1}^{3} \left[\frac{dx_i}{dt}\right]^2}\, dt.$$



Define its arc length function s by

$$s = s(t) = \int_a^t |r'(u)|\, du = \int_a^t \sqrt{\sum_{i=1}^{3} \left[\frac{dx_i}{du}\right]^2}\, du.$$

Then $s(t)$ is the length of the part of C between $r(a)$ and $r(t)$, and, by the Fundamental Theorem of Calculus, $s(t)$ is differentiable with

$$\frac{ds}{dt} = |r'(t)| \quad \text{for } t \in (a, b).$$

It is often useful to parametrize a curve with respect to arc length because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system or a particular parametrization.

Examples

1. Find the length of the arc of the circular helix with vector equation $r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.
2. Reparametrize the helix $r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t.

Definition A parametrization $r(t)$ is called smooth (or regular) on an interval I if $r'(t)$ is continuous and $r'(t) \neq 0$ for each $t \in I$.

Remark Let C be a smooth curve defined by the vector function $r(t)$, $t \in [a, b]$, $s(t) = \int_a^t |r'(u)|\, du$, and let $L = \int_a^b |r'(u)|\, du$ be the length of C.

Since $\frac{ds}{dt} = |r'(t)| > 0$ for $t \in (a, b)$, $s = s(t) : [a, b] \to [0, L]$ is an 1-1, increasing function and has a differentiable inverse $t = t(s) : [0, L] \to [a, b]$ such that

$$1 = \frac{dt}{dt} = \frac{dt}{ds}\frac{ds}{dt} \implies \frac{dt}{ds} = \frac{1}{ds/dt} = \frac{1}{|r'(t)|}.$$

A curve is called smooth if it has a smooth parametrization. A smooth curve has no sharp corners (e.g. $r(t) = (t, |t|)$, $t \in \mathbb{R}$, at $(0, 0)$, $r'(0)$ does not exist) or cusps (e.g. $r(t) = (t^2, t^3)$, $t \geq 0$, at $(0, 0)$, $r'(0) = (0, 0)$); when the tangent vector turns, it does so continuously. If C is a smooth curve defined by the vector function r, recall that the unit tangent vector $T(t)$ of C at $r(t)$ is given by

$$T(t) = \frac{r'(t)}{|r'(t)|} \implies \langle T(t), T(t)\rangle = 1 \implies \frac{d}{dt}\langle T(t), T(t)\rangle = 0 \implies \langle T'(t), T(t)\rangle = 0 \text{ for all } t \in I$$

and $T(t)$ indicates the direction of the curve.

Definition Let C be a smooth (or regular) curve defined by the vector function $r(t)$, $t \in I$, and let $T(t)$ be the unit tangent vector to C at $r(t)$. Then the curvature of a curve at $r(t)$ is

$$k = \left|\frac{dT}{ds}\right|.$$

The curvature is easier to compute if it is expressed in terms of the parameter t instead of s, so we use the Chain Rule to write

$$\frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt} \quad \text{and} \quad k = \left|\frac{dT}{ds}\right| = \frac{|T'(t)|}{|s'(t)|} = \frac{|T'(t)|}{|r'(t)|}.$$

Example Show that the curvature of a circle of radius a is $\frac{1}{a}$.

Definition Let C be a smooth (or regular) curve defined by the vector function $r(t)$, $t \in I$, and let $T(t)$ be the unit tangent vector to C at $r(t)$. Suppose that the curvature $k(t) \neq 0 \iff T'(t) \neq 0$ at $r(t) \in C$. Then we can define the principal unit normal vector $N(t)$ (or simply unit normal) as

$$N(t) = \frac{T'(t)}{|T'(t)|} \quad \text{when } k(t) \neq 0 \implies \frac{dT}{ds} = k(s)N(s), \quad \text{where } s = s(t) \text{ is the arc length parameter}$$

and the vector

$$B(t) = T(t) \times N(t)$$

is called the binormal vector.

Note that $B(t)$ is perpendicular to both $T(t)$ and $N(t)$ and is also a unit vector, i.e. $\langle B(t), T(t)\rangle = 0$, $\langle B(t), N(t)\rangle = 0$ and $|B(t)| = |T(t) \times N(t)| = |T(t)|\,|N(t)| \sin(\pi/2) = 1$.

Theorem The curvature of the curve given by the vector function $r(t)$ is

$$k(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

Proof Since

$$k(s) = |k(s)B(s)| = |k(s)\,T(s) \times N(s)| = |T(s) \times \frac{dT}{ds}| = |\frac{dr}{ds} \times \frac{d^2r}{ds^2}|,$$

and since $T(s) = \frac{dr}{ds}$,

$$\frac{dr}{ds} = r'(t)\frac{dt}{ds} = \frac{r'(t)}{|r'(t)|} \quad \text{and} \quad \frac{d^2r}{ds^2} = \frac{r''(t)}{|r'(t)|^2} + r'(t)\frac{d^2t}{ds^2} \overset{r'\times r'=0}{\implies} \frac{dr}{ds} \times \frac{d^2r}{ds^2} = \frac{r'(t) \times r''(t)}{|r'(t)|^3},$$

we have

$$k(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}.$$

Remark Let C be a smooth (or regular) curve defined by the position vector function $r(t)$, $t \in I$, and let $s = s(t)$ be the arc length parameter. Since

$$\frac{dr}{ds} = T(s), \quad \frac{d^2r}{ds^2} = \frac{r''(t)}{|r'(t)|^2} + r'(t)\frac{d^2t}{ds^2} \quad \text{and} \quad \frac{dT}{ds} = k(s)N(s),$$

$$\Longrightarrow \quad k(s)N(s) = \frac{d^2r}{ds^2} = \frac{r''(t)}{|r'(t)|^2} + r'(t)\frac{d^2t}{ds^2} = \frac{a(t)}{|v(t)|^2} + T(s)\left(|v(t)|\,\frac{d^2t}{ds^2}\right)$$

$$\overset{N\perp T,\,|N|=|T|=1}{\Longrightarrow} \quad a_N(t) = \langle a(t)\,,\, N(s)\rangle = k(s)|v(t)|^2, \;\; a_T(t) = \langle a(t)\,,\, T(s)\rangle = -|v(t)|^3 \cdot \frac{d^2t}{ds^2}$$

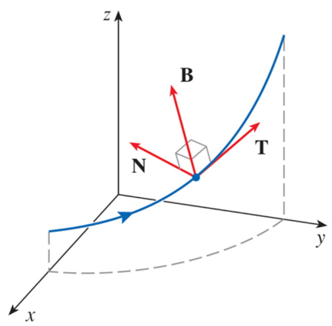
where $v(t) = r'(t)$, $a(t) = r''(t)$ are respectively the velocity, acceleration vectors at the position $r(t)$ and $a_N(t)$, $a_T(t)$ are the normal and tangent components of $a(t)$.

Example For the special case of a plane curve with equation $y = f(x)$, we choose x as the parameter and write $r(x) = x\mathbf{i} + f(x)\mathbf{j}$. Then $r'(x) = \mathbf{i} + f'(x)\mathbf{j}$, $r''(x) = f''(x)\mathbf{j}$, and

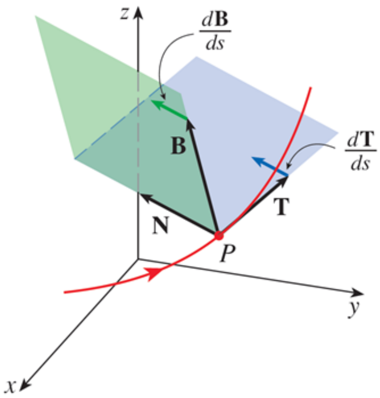
$$k(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}.$$

Example Find the unit normal and binormal vectors for the helix $r(t) = \langle \cos t, \sin t, t\rangle$.

Solution At $r(t)$, since $r'(t) = \langle -\sin t, \cos t, 1\rangle$, the unit tangent vector $T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{2}}\langle -\sin t, \cos t, 1\rangle$, $T'(t) = \frac{1}{\sqrt{2}}\langle -\cos t, -\sin t, 0\rangle$, the unit normal vector $N(t) = \frac{T'(t)}{|T'(t)|} = \langle -\cos t, -\sin t, 0\rangle$, and the unit binormal vector $B(t) = T(t) \times N(t) = \frac{1}{\sqrt{2}}\langle \sin t, -\cos t, 1\rangle$.



Definition The plane determined by the vectors N and B at a point P on a curve C is called the normal plane of C at P. It consists of all lines that are orthogonal to the tangent vector T. The plane determined by the vectors T and N is called the osculating plane of C at P.



Definition The circle of curvature, or the osculating circle, of C at P is the circle in the osculating plane that passes through P with radius $\frac{1}{k}$ and center a distance $\frac{1}{k}$ from P along the vector N. The center of the circle is called the center of curvature of C at P.

Remarks

- We can think of the circle of curvature as the circle that best describes how C behaves near P, it shares the same tangent, normal, and curvature at P.
- Note that the curvature $k = |dT/ds|$ at a point P on a curve C indicates how tightly the curve "bends." Since T is a normal vector for the normal plane, dT/ds tells us how the normal plane changes as P moves along C.
- Since B is normal to the osculating plane, dB/ds gives us information about how the osculating plane changes as P moves along C. Thus there is a scalar τ such that

$$\frac{dB}{ds} = \tau N \implies \tau = \langle \frac{dB}{ds}, N \rangle \iff \tau(t) = \frac{\langle B'(t), N(t) \rangle}{|r'(t)|},$$

  where τ is called the torsion of C at $P = r(t)$.

Example Find the torsion $\tau(t)$ of the helix $r(t) = \langle \cos t, \sin t, t \rangle$ at $r(t)$.

Solution At $r(t)$, since $r'(t) = \langle -\sin t, \cos t, 1 \rangle$, $|r'(t)| = \sqrt{2}$,
$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{2}}\langle -\sin t, \cos t, 1 \rangle$, $T'(t) = \frac{1}{\sqrt{2}}\langle -\cos t, -\sin t, 0 \rangle$,
$N(t) = \frac{T'(t)}{|T'(t)|} = \langle -\cos t, -\sin t, 0 \rangle$, $B(t) = T(t) \times N(t) = \frac{1}{\sqrt{2}}\langle \sin t, -\cos t, 1 \rangle$, we obtain that
$B'(t) = \frac{1}{\sqrt{2}}\langle \cos t, \sin t, 0 \rangle$ and
$\tau(t) = \frac{\langle B'(t), N(t) \rangle}{|r'(t)|} = \frac{\langle B'(t), N(t) \rangle}{\sqrt{2}} = -\frac{\cos^2 t + \sin^2 t}{2} = -\frac{1}{2}$.

More Facts (補充教材)

Definitions Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be vectors in $\mathbb{R}^3$.

(a) The inner product $\langle x\,,\,y\rangle = x \cdot y$ is defined by

$$\langle x\,,\,y\rangle = x \cdot y = \sum_{i=1}^{3} x_i y_i.$$

Note that for any x, y, $z \in \mathbb{R}^3$ and for any $a \in \mathbb{R}$, the definition implies that

- $\langle x\,,\,y\rangle = x \cdot y = \sum_{i=1}^{3} x_i y_i = \sum_{i=1}^{3} y_i x_i = y \cdot x = \langle y\,,\,x\rangle$ (inner product is symmetric),
- $\langle z\,,\,ax + y\rangle = \langle ax + y\,,\,z\rangle = a\langle x\,,\,z\rangle + \langle y\,,\,z\rangle$ (inner product is bilinear).

(b) The cross product $x \times y$ is defined by ,

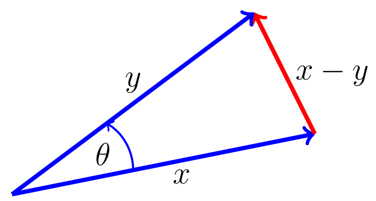
$$x \times y = \left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, -\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right) = (x_2y_3 - x_3y_2\,,\,x_3y_1 - x_1y_3\,,\,x_1y_2 - x_2y_1)\,.$$

The definition implies that

$$y \times x = \left(\begin{vmatrix} y_2 & y_3 \\ x_2 & x_3 \end{vmatrix}, -\begin{vmatrix} y_1 & y_3 \\ x_1 & x_3 \end{vmatrix}, \begin{vmatrix} y_1 & y_2 \\ x_1 & x_2 \end{vmatrix} \right) = -\left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, -\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right) = -x \times y,$$

Remarks

(a) By the law of cosines,



$$|x-y|^2 = (\text{length of the vector}\, x-y)^2 = |x|^2 + |y|^2 - 2|x||y|\cos\theta.$$

This gives

$$\begin{aligned} 2|x||y|\cos\theta &= |x|^2 + |y|^2 - |x-y|^2 \\ &= \sum_{i=1}^{3} x_i^2 + \sum_{i=1}^{3} y_i^2 - \sum_{i=1}^{3} (x_i - y_i)^2 \\ &= 2\sum_{i=1}^{3} x_i y_i \\ &= 2\langle x\,,\,y\rangle. \end{aligned}$$

Hence we have $\sum_{i=1}^{3} x_i y_i = \langle x\,,\,y\rangle = |x||y|\cos\theta$, where θ is the angle between x and y.

(b) Let $z = (z_1, z_2, z_3)$ be a vector in $\mathbb{R}^3$. Then

$$\begin{aligned}
\langle z\,,\, x\times y\rangle &= \left\langle z\,,\, \left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3\end{vmatrix}\,,\, -\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3\end{vmatrix}\,,\, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2\end{vmatrix}\right)\right\rangle \\
&= z_1\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3\end{vmatrix} - z_2\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3\end{vmatrix} + z_3\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2\end{vmatrix} \\
&= \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3\end{vmatrix} = \det(z, x, y) = \text{determinant of } \begin{pmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3\end{pmatrix}
\end{aligned}$$

For any a, $b \in \mathbb{R}$, since

$$\langle ax+by\,,\, x\times y\rangle = \begin{vmatrix} ax_1+by_1 & ax_2+by_2 & ax_3+by_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3\end{vmatrix} = a\begin{vmatrix} x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3\end{vmatrix} + b\begin{vmatrix} y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3\end{vmatrix} = 0,$$

If $x\times y \neq 0 \in \mathbb{R}^3$, then $x\times y$ is a vector perpendicular to the plane spanned by x and y.

(c) If x and y are parallel, then $x\times y = 0 \in \mathbb{R}^3$.

(d) Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. Since

$$\langle e_i, e_j\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \quad \text{and } e_i\times e_i = 0 \quad \text{for } i = 1,\,2,\,3,$$

$$e_1\times e_2 = e_3,\;\; e_1\times e_3 = -e_2,\;\; e_2\times e_3 = e_1 \text{ and } \det(e_i\times e_j, e_i, e_j) = 1 \text{ for all } 1\le i\neq j\le 3,$$

we have

$$\begin{aligned}
(e_i\times e_j)\cdot(e_k\times e_\ell) &= \det(e_i\times e_j, e_k, e_\ell) \\
&= \begin{cases} 0 & \text{if } i=j, \text{ or } k=\ell, \text{ or } \{i\neq j\}\neq\{k\neq\ell\}, \\ 1 & \text{if } \{i=k\neq j=\ell\} \\ -1 & \text{if } \{i=\ell\neq j=k\} \end{cases} \\
&= \begin{vmatrix} e_i\cdot e_k & e_j\cdot e_k \\ e_i\cdot e_\ell & e_j\cdot e_\ell\end{vmatrix} \quad \text{for all } i,\, j,\, k,\, \ell = 1,\,2,\,3,
\end{aligned}$$

and the identity

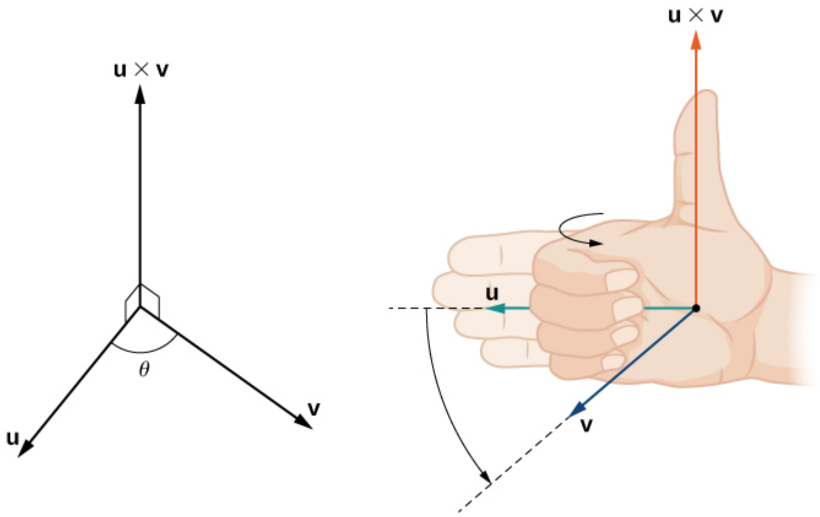
$$(u\times v)\cdot(x\times y) = \begin{vmatrix} u\cdot x & v\cdot x \\ u\cdot y & v\cdot y\end{vmatrix} \quad \text{for all } u,\, v,\, x,\, y\in\mathbb{R}^3$$

by noting that both sides are linear in $u = \sum_{i=1}^{3} u_i e_i$, $v = \sum_{j=1}^{3} v_j e_j$, $x = \sum_{k=1}^{3} x_k e_k$, $y = \sum_{\ell=1}^{3} y_\ell e_\ell$.

In particular, for any x, $y\in\mathbb{R}^3$ this implies that

$$(x\times y)\cdot(x\times y) = \begin{vmatrix} x\cdot x & y\cdot x \\ x\cdot y & y\cdot y\end{vmatrix} = (x\cdot x)^2\,(y\cdot y)^2 - (x\cdot y)^2 = |x|^2|y|^2 - |x|^2|y|^2\cos^2\theta = |x|^2|y|^2\sin^2\theta$$

so $|x\times y| = |x|\,|y|\,\sin\theta$, where θ is the angle between x and y. Hence $|x\times y| =$ the area of the parallelogram spanned by x and y.



(e) If $x \times y \neq 0 \in \mathbb{R}^3$, then $\det(x \times y, x, y) = \langle x \times y\,,\, x \times y\rangle = |x \times y|^2 > 0$, and x, y and $x \times y$ form a right-handed triple.

Remarks Let $x,\ y : (a, b) \to \mathbb{R}^3$ be differentiable functions defined on (a, b) with vector value in $\mathbb{R}^3$. Then

(a) $\dfrac{d}{dt}\langle x\,,\, y\rangle = \langle \dfrac{dx}{dt}\,,\, y\rangle + \langle x\,,\, \dfrac{dy}{dt}\rangle$.

Proof For each $t \in (a, b)$, let

$$x = x(t) = (x_1(t), x_2(t), x_3(t)) = (x_1, x_2, x_3),\ \ y = y(t) = (y_1(t), y_2(t), y_3(t)) = (y_1, y_2, y_3).$$

Since $\langle x\,,\, y\rangle = \sum_{i=1}^{3} x_i y_i$, we have

$$\frac{d}{dt}\langle x\,,\, y\rangle = \sum_{i=1}^{3} \frac{d}{dt}(x_i y_i) = \sum_{i=1}^{3} \frac{dx_i}{dt} y_i + \sum_{i=1}^{3} \frac{dy_i}{dt} x_i = \langle \frac{dx}{dt}\,,\, y\rangle + \langle x\,,\, \frac{dy}{dt}\rangle.$$

(b) $\dfrac{d}{dt}\big(x \times y\big) = \dfrac{dx}{dt} \times y + x \times \dfrac{dy}{dt}$.

(c) If $|x(t)| = r$, a positive constant, for all $t \in (a, b)$, then $\dfrac{d}{dt}\langle x\,,\, x\rangle = \dfrac{d}{dt}(r^2) = 0$. We have $\langle \dfrac{dx}{dt}\,,\, x\rangle = 0$ for all $t \in (a, b) \Longrightarrow \dfrac{dx}{dt} \perp x$ (互相垂直) whenever $\dfrac{dx}{dt} \neq 0$.